

Locally strongly transitive automata and the Hybrid Cerny-Road coloring problem

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Abstract. An independent system of words for a finite automaton is a set of k words taking any state s into k distinct states which do not depend on s . We present some recent application of independent sets to the synchronization problem and to synchronizing colorings of aperiodic graphs. In particular, we prove that if an aperiodic, strongly connected digraph of constant outdegree with n vertices has an Hamiltonian path, then it admits a synchronizing coloring with a reset word of length $2(n - 2)(n - 1) + 1$.

An important concept in Computer Science is that of *synchronizing automaton*. A deterministic automaton is called *synchronizing* if there exists an input-sequence, called *synchronizing* or *reset word*, such that the state attained by the automaton, when this sequence is read, does not depend on the initial state of the automaton itself. Two fundamental problems which have been intensively investigated in the last decades are based upon this concept: the *Černý conjecture* and the *Road coloring problem*.

The Černý conjecture [8] claims that a deterministic synchronizing n -state automaton has a reset word of length not larger than $(n - 1)^2$. This conjecture has been shown to be true for several classes of automata (cf. [2-4, 6-8, 10-14, 17]). The interested reader is referred to [17] for a historical survey of the Černý conjecture and to [5] for synchronizing unambiguous automata. In this theoretical setting, two results recently proven in [7] and [4] respectively, are relevant to us.

In [7], the authors have introduced the notion of *independent set of words*.

Definition 1. Let $\mathcal{A} = \langle Q, A, \delta \rangle$ be an automaton. A set of k words $W = \{w_0, \dots, w_{k-1}\}$ is called *independent* if there exist k distinct states $q_0, \dots, q_{k-1}$ of $\mathcal{A}$ such that, for all $s \in Q$,

$$\{\delta(s, w_0), \dots, \delta(s, w_{k-1})\} = \{q_0, \dots, q_{k-1}\}.$$

The set $R = \{q_0, \dots, q_{k-1}\}$ will be called the *range* of W .

An automaton is called *locally strongly transitive* if it has an independent set of words. The main result of [7] is that any synchronizing locally strongly transitive n -state automaton has a reset word of length not larger than $(k-1)(n+L_W)+\ell_W$, where k is the cardinality of an independent set W and L_W and ℓ_W denote respectively the maximal and the minimal length of the words of W . In the case where all the states of the automaton are in the range, the automaton $\mathcal{A}$ is said to be *strongly transitive*. Strongly transitive automata have been studied in [6]. This notion is related with that of regular automata introduced in [14]. A remarkable example of locally transitive automata is that of *1-cluster automata*, recently investigated in [4]. A n -state automaton is called 1-cluster if there exists a letter a such that the graph of the automaton has a unique cycle labelled by a power of a . Indeed, denoting by k the length of the cycle, one easily verifies that the words

$$a^{n-1}, a^{n-2}, \dots, a^{n-k}$$

form an independent set of the automaton whose range is the set of vertices of the cycle. In [4] it is proven that every 1-cluster synchronizing n -state automaton has a reset word of length not larger than $2(n-1)(n-2)+1$. Let $\mathcal{A} = \langle Q, A, \delta \rangle$ be a n -state automaton. We say that a set of states K of $\mathcal{A}$ is *reducible* if, for some word w , $\delta(K, w)$ is a singleton. Given two states p, q of $\mathcal{A}$, we say that the pair (p, q) is *stable* if, for all $u \in A^*$, there exists $v \in A^*$ such that $\delta(p, uv) = \delta(q, uv)$. The set of stable pairs is a congruence of the automaton $\mathcal{A}$, which is called *stability relation*. This congruence, introduced in [9], plays a fundamental role in the solution [15] of the Road coloring problem. It is easily seen that an automaton is synchronizing if and only if the stability relation is the universal equivalence. A set $K \subseteq Q$ is *stable* if for any $p, q \in K$, the pair (p, q) is stable.

It is worth mentioning that any stable set K is reducible. Thus, even if $\mathcal{A}$ is not synchronizing, one may want to evaluate the minimal length of a word w such that $\text{Card}(\delta(K, w)) = 1$. The main result on this point is the following.

Theorem 1. *Let $\mathcal{A}$ be a n -state automaton with an independent set W . If $\mathcal{A}$ is not synchronizing, then for any stable set K there exists a word v such that*

$$\text{Card}(\delta(K, v)) = 1, \quad |v| \leq \left(\frac{\text{Card}(W)}{2} - 1 \right) (n + L_W - 1) + L_W,$$

where L_W denotes the maximal length of the words of W .

Since any 1-cluster n -state automaton has an independent set W with $L_W = n - 1$, taking into account that $\text{Card}(W) \leq n$, the following result follows from Theorem 1.

Corollary 1. *Let K be a stable set of a 1-cluster n -state automaton which is not synchronizing. There exists a word v such that $\text{Card}(\delta(K, v)) = 1$ and $|v| \leq (n - 1)^2$.*

In the case where $\mathcal{A}$ is synchronizing, then Q itself is a stable set. With some minor changes in the proof of Theorem 1, one obtains the following result which refines the quoted bound of [7].

Theorem 2. *Any synchronizing n -state automaton with an independent set W has a reset word of length*

$$(\text{Card}(W) - 1)(n + L_W - 1) + \ell_W,$$

where L_W and ℓ_W denote respectively the maximal and the minimal length of the words of W .

The second problem we have mentioned above is the *Road coloring problem* which concerns the study of the synchronizing colorings of aperiodic graphs. In the sequel, with the word graph, we will term a finite, directed multigraph with all vertices of the same outdegree. A graph is *aperiodic* if the greatest common divisor of the lengths of all cycles of the graph is 1. A graph is called an *AGW-graph* if it is aperiodic and strongly connected. A synchronizing automaton which is obtained by a labeling of the edges of a graph G will be called a *synchronizing coloring* of G . The *Road coloring problem* asks for the existence of a synchronizing coloring for every AGW-graph. This problem was formulated in the context of Symbolic Dynamics by Adler, Goodwyn and Weiss and it is explicitly stated in [1]. In 2007, Trahtman has positively solved this problem [15]. Trahtman's solution has electrified the community of formal language theories and recently Volkov has raised the following problem [16].

Hybrid Černý–Road coloring problem. *Let G be an AGW-graph. What is the minimum length of a reset word for a synchronizing coloring of G ?*

It is worth to mention that Ananichev has found, for any $n \geq 2$, an AGW-graph of n vertices such that the length of the shortest reset word for any synchronizing coloring of the graph is $(n - 1)(n - 2) + 1$ (see [16]). In [7], the authors have proven that, given an AGW-graph G of n vertices, without multiple edges, such that G has a simple cycle of prime length $p < n$, there exists a synchronizing coloring of G with a reset word of length $(2p - 1)(n - 1)$. Moreover, in the case $p = 2$, that is, if G contains a cycle of length 2, then, also in presence of multiple edges, there exists a synchronizing coloring with a reset word of length $5(n - 1)$. Here, we continue the investigation of the Hybrid Černý–Road coloring problem on a very natural class of digraphs, those having a *Hamiltonian path*. The new theorem we prove is the following.

Theorem 3. *Let G be an AGW-graph with $n > 1$ vertices. If G has a Hamiltonian path, then there is a synchronizing coloring of G with a reset word w of length*

$$|w| \leq 2(n - 2)(n - 1) + 1.$$

The proof of Theorem 3 is based upon Corollary 1 and upon some techniques and results on synchronizing colorings of graphs.

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